1. Show that (𝐴∪𝐶)∩(𝐵∪𝐶)⊆(𝐴∩𝐵)∪𝐶

Proof: For any , we have . There are two possibilities, and .

If , then by the definition of union.

If , then implies , and implies . Hence , which further implies .

Hence, for both , it holds that . Thus (𝐴∪𝐶)∩(𝐵∪𝐶)⊆(𝐴∩𝐵)∪𝐶.

1. Write each of the followings explicitly

a).

b).

1. Let . Show that the following relation is an equivalence relation on : if and only if .

Proof: Reflexive: since , we know that for any . Hence is reflexive.

Symmetric: for any , we know , hence , which implies . Hence is symmetric.

Transitive: if , , then we know that , and . Hence , which means . Hence, is transitive.

1. Let and be any two partial orders on the same set . Show that is a partial order.

Proof: Reflexive: for any , we have and since and are reflexive. Hence, , which means is reflexive.

Anti-symmetric: if , then we know and . Since and are antisymmetric, we have and . Hence , which means is anti-symmetric.

Transitive: if , , then we have , and , . Since and are both transitive, we have and . Hence , which means is transitive.

1. Show that any function from a finite set to itself contains a cycle.

Proof: Let the function be for some finite set . Since is a finite set, let for some .

We define functions , and for and .

Pick an arbitrary , and consider the sequence of elements . Since , these elements can take at most distinct values. By Pigeonhole principle, there exist some such that . Hence, vertices form a cycle.

1. Show that in any group of at least two people there are at least two persons that have the same number of acquaintances within the group. (Assuming acquaintance is bi-direction).

Proof: Suppose on the contrary that there exists a group of people such that any two people have different number of acquaintances within the group. Index the people arbitrarily as , and let the number of acquaintances of be . By our assumption for any .

We first claim that , that is, is a permutation of . Suppose the claim is not true, then only takes at most distinct values. Then by pigeonhole principle there exist some such that , which is a contradiction to our assumption. Hence, the claim is true, .

According to the claim above, there exists some person, say, who has no acquaintance, and some person, say, who knows everyone else. However, this is impossible since acquaintance is bi-directional, the person who knows everyone else should also know , which means knows at least one other person, contradicting to the fact that has no acquaintance. Hence, there does not exist a group of people such that any two people have different number of acquaintances within the group, i.e., in any group of at least two people there are at least two persons that have the same number of acquaintances within the group.

1. \* Show that there is no bijection between and .

This is really a generalization of Cantor’s proof, given above. Suppose that there really is a bijection f : S → 2S. We create a new set A as follows. We say that A contains the element s ∈ S if and only if s is not a member of f(s). This makes sense, because f(s) is a subset of S.

Since A is a subset of S, we have A = f(a) for some a ∈ S. If a ∈ A then a ∈ f(a). But then, by deﬁnition, a is not a member of A. On the other hand, if a 6∈ A, then a ∈ f(a). But, again, this is a contradiction. The only way out of the contradiction is to realize that there can be no bijection f. We can start with S0 = N, and recursively deﬁne Sn = 2Sn−1. That is, Sn is the set of subsets of Sn−1. Then, the sets S0,S1,S2,... form an inﬁnite heirarchy of sets, each one so much larger than the previous one that there is no bijection between it and the previous one. The fun doesn’t stop there. We can deﬁne

Σ0 =

∞ [ n=0

Sn.

Then, there is no bijection between Σ0 and Sn for any n. The set Σ0 is larger than all of the sets previously deﬁned. One can now deﬁne Σn = 2Σn−1. And so on.